

TROPICAL MATRICES AND GROUP REPRESENTATIONS

YAROSLAV SHITOV

ABSTRACT. The paper gives a complete description of the subgroups of the semigroup of tropical n -by- n matrices up to an isomorphism. In particular, we show that every of these groups has a torsion-free abelian subgroup of index at most $n!$, proving the conjecture of Johnson and Kambites.

1. INTRODUCTION

The set \mathbb{R} of reals extended by adding an infinite negative element $-\infty$ is called the *tropical semiring* and is also known as the *max-plus algebra*. The tropical arithmetic operations on $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ are $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$. The main object of our research is the set $\overline{\mathbb{R}}^{n \times n}$ of tropical n -by- n matrices. We are interested in studying the multiplicative structure of tropical matrices. The multiplication of such matrices is defined as ordinary matrix multiplication with $+$ and \cdot replaced by the tropical operations \oplus and \otimes .

The study of linear algebra over the tropical semiring is important for many different applications (see [1, 5]). There is a number of purely linear-algebraic important problems for tropical matrices, for example, the eigenvalues and eigenvectors problem, the problem of solving linear systems, computational problems for the rank functions, see [3, 5]. Another important approach considers the set of tropical matrices from the point of view of the semigroup theory. The paper [4] is devoted to the solution of the Burnside-type problem for semigroups of tropical matrices. Johnson and Kambites in the recent paper [8] have developed the study of the semigroup-theoretic structure of tropical matrices. They consider Green's relations on the semigroup $(\overline{\mathbb{R}}^{n \times n}, \otimes)$, groups of tropical matrices, and idempotent tropical matrices. They give a complete description of the subgroups of $(\overline{\mathbb{R}}^{n \times n}, \otimes)$ in the case when $n = 2$. The study of Green's relations on the semigroup of tropical matrices has been developed in [6, 7]. In [6], the complete description of the \mathcal{D} -relation has been provided. In [7], the important characterization of the \mathcal{J} -order has been given, and the connection of Green's relations with the rank functions of tropical matrices has been studied.

The aim of our paper is to solve the problem that has arisen from the paper [8]. Namely, we are interested in a complete characterization of the subgroups of the semigroup $(\overline{\mathbb{R}}^{n \times n}, \otimes)$. We show that every subgroup of the semigroup of tropical n -by- n matrices admits a faithful representation with tropical monomial n -by- n matrices. We prove that every subgroup \mathcal{G} of $(\overline{\mathbb{R}}^{n \times n}, \otimes)$ is isomorphic to a subgroup of the wreath product $\mathbb{R} \wr \mathcal{S}_n$, and, conversely, every subgroup of $\mathbb{R} \wr \mathcal{S}_n$ can be realized with tropical n -by- n matrices. Our results confirm the conjecture proposed in [8] that every group admitting a faithful representation by $n \times n$ tropical matrices must

have a torsion-free abelian subgroup of index at most $n!$. We also give an upper bound for the order of a periodic group of tropical n -by- n matrices, developing the result proven in [2].

Throughout our paper \mathcal{S}_n will denote the symmetric group on $\{1, \dots, n\}$. By $a_{i(\cdot)}$ we denote the i th row of a matrix A , by $A[r_1, \dots, r_k]$ the submatrix of A formed by the rows with numbers r_1, \dots, r_k . We say that a matrix $P \in \overline{\mathbb{R}}^{n \times n}$ is *monomial* if there exists $\sigma = \sigma(P) \in \mathcal{S}_n$ such that $p_{ij} \neq -\infty$ if and only if $i = \sigma(j)$. In this case, P is called *diagonal* if $\sigma(P)$ is the identity. Note that the diagonal matrix with zeros on the diagonal is the neutral element of the semigroup $(\overline{\mathbb{R}}^{n \times n}, \otimes)$.

2. SUBGROUPS OF THE SEMIGROUP $(\overline{\mathbb{R}}^{n \times n}, \otimes)$

We need the concept of the row rank (see [1]) of a tropical matrix for our considerations.

Definition 2.1. A tropical matrix $B \in \overline{\mathbb{R}}^{n \times m}$ is said to be of *full row rank* if no row of A can be expressed as a linear combination of other rows, that is, the condition

$$(2.1) \quad b_{i(\cdot)} = \bigoplus_{k \in \{1, \dots, n\} \setminus \{i\}} \lambda_k \otimes b_{k(\cdot)}$$

fails to hold for every $i \in \{1, \dots, n\}$ and $\lambda_1, \dots, \lambda_n \in \overline{\mathbb{R}}$.

We also need the following lemma.

Lemma 2.2. Let $A, B, C, D \in \overline{\mathbb{R}}^{n \times n}$ be such that $B = C \otimes A$, $A = D \otimes B$. If the row rank of B is full, then there exists a monomial matrix $P \in \overline{\mathbb{R}}^{n \times n}$ such that $B = P \otimes A$.

Proof. Since $B = C \otimes A$ and $A = D \otimes B$ imply that $B = C \otimes D \otimes B$, we have

$$(2.2) \quad b_{i(\cdot)} = \bigoplus_{p=1}^n \left[\left(\bigoplus_{k=1}^n c_{ik} \otimes d_{kp} \right) \otimes b_{p(\cdot)} \right].$$

If $\bigoplus_{k=1}^n c_{ik} \otimes d_{ki} < 0$, then the summand of $p = i$ can be omitted from the right-hand side of (2.2). In this case, the condition (2.1) is satisfied, so the row rank of B is not full. The contradiction shows that indeed $\bigoplus_{k=1}^n (c_{ik} \otimes d_{ki}) \geq 0$.

Then for some $\kappa = \kappa(i)$, we have $c_{i\kappa} \otimes d_{\kappa i} \geq 0$. From $B = C \otimes A$ it follows that $b_{it} \geq c_{i\kappa} \otimes a_{\kappa t}$ for every $t \in \{1, \dots, n\}$. On the other hand,

$$c_{i\kappa} \otimes a_{\kappa t} = c_{i\kappa} \otimes \left(\bigoplus_{p=1}^n d_{\kappa p} \otimes b_{pt} \right) \geq c_{i\kappa} \otimes d_{\kappa i} \otimes b_{it} \geq b_{it},$$

so indeed $b_{it} = c_{i\kappa} \otimes a_{\kappa t}$ for every $t \in \{1, \dots, n\}$. Thus every row of B is some row of A multiplied by a scalar. Since the row rank of B is full, the result follows. \square

The following lemma deals with matrices whose row rank is not full.

Lemma 2.3. Let \mathcal{G} be a subgroup of $(\overline{\mathbb{R}}^{n \times n}, \otimes)$, $n \geq 2$. If the row rank of some matrix A from \mathcal{G} is not full, then \mathcal{G} admits a faithful representation with tropical $(n-1)$ -by- $(n-1)$ matrices.

Proof. By Definition 2.1, some row of A is a linear combination of other its rows. So for some $i \in \{1, \dots, n\}$ we have $A = P \otimes \overline{A}$, where the matrix $\overline{A} \in \mathbb{R}^{(n-1) \times n}$ is obtained from A by removing the i th row, and $P \in \mathbb{R}^{n \times (n-1)}$ is such that the matrix $P[1, \dots, i-1, i+1, \dots, n]$ is the neutral element of $(\mathbb{R}^{(n-1) \times (n-1)}, \otimes)$. Since \mathcal{G} is a group, for every $G \in \mathcal{G}$ there exists $B \in \mathcal{G}$ such that $G = A \otimes B$.

Thus we see that $G = P \otimes \overline{G}$. The matrix \overline{G} here is uniquely determined, because $P[1, \dots, i-1, i+1, \dots, n]$ is the neutral element. The map φ sending $G \in \mathcal{G}$ to $\overline{G} \otimes P \in \mathbb{R}^{(n-1) \times (n-1)}$ is therefore well defined.

Note that for every $G, H \in \mathcal{G}$ it holds that

$$\varphi(G \otimes H) = \varphi(P \otimes \overline{G} \otimes P \otimes \overline{H}) = \overline{G} \otimes P \otimes \overline{H} \otimes P = \varphi(G) \otimes \varphi(H),$$

so φ is a homomorphism. Moreover, if $\varphi(G) = \varphi(H)$, then $\overline{G} \otimes P = \overline{H} \otimes P$, so in this case $P \otimes \overline{G} \otimes P \otimes \overline{G} = P \otimes \overline{H} \otimes P \otimes \overline{G}$, or $G \otimes G = H \otimes G$. Since \mathcal{G} is a group, the condition $\varphi(G) = \varphi(H)$ therefore implies that $G = H$, proving that φ is injective. \square

Now we are ready to prove the one of our main results.

Theorem 2.4. *Every subgroup of the semigroup $(\mathbb{R}^{n \times n}, \otimes)$ admits a faithful representation with tropical monomial n -by- n matrices.*

Proof. The case of $n = 1$ is trivial, and we proceed by the induction on n . Let $n \geq 2$, \mathcal{G} be a subgroup of $(\mathbb{R}^{n \times n}, \otimes)$, E be a neutral element of \mathcal{G} . The two cases are then possible.

1. Let \mathcal{G} contain a matrix whose row rank is not full. Lemma 2.3 shows that in this case \mathcal{G} admits a faithful representation with tropical $(n-1)$ -by- $(n-1)$ matrices. The inductive hypothesis then shows that \mathcal{G} has a faithful representation with tropical monomial $(n-1)$ -by- $(n-1)$ matrices, so the result follows.

2. Now let the matrices from \mathcal{G} be of full row rank. In this case, from Lemma 2.2 it follows that for every $G \in \mathcal{G}$ there exists a monomial matrix $\mathcal{P}_G \in \mathbb{R}^{n \times n}$ such that $G = \mathcal{P}_G \otimes E$. Since the row rank of G is full, we see that the matrix \mathcal{P}_G is uniquely determined. So we can define the map ψ sending $G \in \mathcal{G}$ to the monomial matrix \mathcal{P}_G . Clearly, ψ is injective. We also see that for every $G, H \in \mathcal{G}$ it holds that $G \otimes H = \mathcal{P}_G \otimes E \otimes H = \mathcal{P}_G \otimes H = \mathcal{P}_G \otimes \mathcal{P}_H \otimes E$, so ψ is a homomorphism. \square

Johnson and Kambites in [8, Section 4] conjectured that every group admitting a faithful representation by $n \times n$ tropical matrices has a torsion-free abelian subgroup of index at most $n!$. Now we are ready to prove this conjecture.

Theorem 2.5. *Let a group \mathcal{G} admit a faithful representation by $n \times n$ tropical matrices. Then \mathcal{G} has a torsion-free abelian subgroup of index at most $n!$.*

Proof. By Theorem 2.4, we assume without a loss of generality that \mathcal{G} consists of tropical monomial n -by- n matrices. Consider the subgroup D of all diagonal matrices from \mathcal{G} . Clearly, D is normal in \mathcal{G} , abelian and torsion-free. It remains to note that matrices $A, B \in \mathcal{G}$ belong to the same coset of D in \mathcal{G} if and only if $\sigma(A) = \sigma(B)$. \square

D'Alessandro and Pasku have shown that every periodic finitely generated subgroup of $(\mathbb{R}^{n \times n}, \otimes)$ is finite, see [2, Proposition 5]. Theorem 2.4 allows us to derive

a more precise characterization. Recall that a group H is *periodic* if each element of H has finite order.

Corollary 2.6. *The order of any periodic subgroup of the semigroup $(\overline{\mathbb{R}}^{n \times n}, \otimes)$ is at most $n!$.*

Proof. By definition, any torsion-free subgroup of a periodic group is trivial. So the result follows from Theorem 2.5. \square

Finally, we note that the group of all tropical monomial n -by- n matrices is isomorphic to the wreath product $\mathbb{R} \wr \mathcal{S}_n$. This gives the following group-theoretic description of the subgroups of $(\overline{\mathbb{R}}^{n \times n}, \otimes)$.

Theorem 2.7. *A group \mathcal{G} admits a faithful representation with tropical n -by- n matrices if and only if \mathcal{G} is isomorphic to a subgroup of the wreath product $\mathbb{R} \wr \mathcal{S}_n$.*

Proof. Follows from Theorem 2.4. \square

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MOSCOW STATE UNIVERSITY, LENINSKIE GORY, 119991, GSP-1, MOSCOW, RUSSIA
E-mail address: yaroslav-shitov@yandex.ru